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On the Structure of Mandelbrot's Percolation Process and Other Random Cantor Sets

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We consider generalizations of Mandelbrot's percolation process. For the process which we call the random Sierpinski carpet, we show that it passes through several different phases as its parameter increases from zero to one. The final section treats the percolation phase.

KEY WORDS: Percolation process; random fractal set; random substitution; Hausdorff dimension.

In his book *The Fractal Geometry of Nature*, Benoit Mandelbrot considers several random sets with a statistically self-similar structure (Mandelbrot[(7)]). One of these, named canonical curdling, has been analyzed by Chayes *et al.*,[(1)] who called this process Mandelbrot percolation. Another example is Mandelbrot's implementation of Hoyle's model of galaxies. Both are instances of a type of random sets which we simply call random Cantor sets and which we define by way of random substitutions. See Falconer[(4)] or Dekking and Grimmett[(3)] for equivalent definitions involving random trees, respectively labeled branching processes. An interesting subclass of these random Cantor sets consists of those generated by Bernoulli random substitutions, where the structure of the set may change drastically with changes of the Bernoulli parameter p. Examples are Mandelbrot's percolation process and a process which we call the random Sierpinski carpet. We define several phases for these processes and show, using previous work of Dekking and Grimmett[(3)] and new methods developed in Section 3, that the random Sierpinski carpet passes through all these phases as p tends from 0 to 1. (For Mandelbrot percolation at

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least one phase is missing.) The final section is devoted to an analysis of the percolation phase of these processes, which is more direct and more general than the approach of Chayes *et al.*[1]

1. SUBSTITUTIONS AND CANTOR SETS

Substitutions are maps which assign words (concatenations of symbols) to symbols, and can be useful to describe self-similar sets or generalizations thereof as, e.g., quasicrystals (cf. Dekking[2]). By way of introduction, we show how to generate the classical Cantor set by means of a substitution. Throughout this paper our set of symbols will be $\{0, 1\}$. We define a substitution σ on this set by

$$\sigma(0) = 000, \qquad \sigma(1) = 101$$

Iterates of σ are defined in the obvious way, e.g., $\sigma^2(1) = 101000101$. Let

$$I_k^n = [(k-1)3^{-n}, k3^{-n}]$$

be the kth triadic interval of order n. Let $w^n = \sigma^n(1)$ and let w_k^n be the kth symbol of this word, $k = 1, ..., 3^n$. For $n = 1, 2, ...$ we define a subset A_n of $[0, 1]$ by

$$A_n = \bigcup_k \{I_k^n : w_k^n = 1\}$$

Then (A_n) is a decreasing sequence of compact sets, and $A = \bigcap_n A_n$ is the classical Cantor set.

Obviously, this construction generalizes to Cantor sets in $\mathbb{R}^d$, by arranging the symbols in $\sigma(0)$ and $\sigma(1)$ in d-dimensional cubes. Thus, one can obtain, e.g., the Menger sponge (Mandelbrot[7]). In this paper we restrict ourselves to the case $d = 2$, considering substitutions from $\{0, 1\}$ to the set of $N \times N$ matrices for some positive integer N. The "holes stay holes" principle in the construction of Cantor sets translates to the requirement

$$\sigma(0) = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

Iterating σ a total of n times leads to an $N^n \times N^n$ matrix $W^n = \sigma^n(1)$. Let $I_{kl}^n = I_k^n \times I_l^n$, where $I_k^n = [(k-1)N^{-n}, kN^{-n}]$, for $1 \leqslant k,\ l \leqslant N^n$. We call such a set I_{kl}^n a level-n square. We now consider the sets

$$A_n = \bigcup_{k,l} \{I_{kl}^n : W_{kl}^n = 1\}$$

The A'_n decrease to a compact subset A of the unit square. As an example, let $N=3$, and let σ be defined by

$$\sigma(0)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},\qquad \sigma(1)=\begin{pmatrix}1&1&1\\1&0&1\\1&1&1\end{pmatrix}$$

then $A=A^{\sigma}$ is the Sierpinski carpet.

Our next step is to randomize this construction. We continue to require $\sigma(0)=\mathbf{0}$ (the $N\times N$ matrix with all entries zero), but $\sigma(1)$ will be random, taking values in a set of $N\times N$ 0–1-valued matrices $\{U_1, ..., U_r\}$; specifically, there are positive numbers $p_1, ..., p_r$ with $p_1+\cdots+p_r=1$ such that $\mathbb{P}[\sigma(1)=U_i]=p_i$, for $1\leqslant i\leqslant r$. (We will use the symbol $\mathbb{P}$ for the probability of an event determined by random substitutions. The probability space associated with this $\mathbb{P}$ will only lurk in the background.)

Iteration is defined as follows: $\sigma^{n+1}(1)$ is obtained by replacing all 0's in $\sigma^n(1)$ by $\sigma(0)=\mathbf{0}$, and all 1's by independent random matrices distributed as $\sigma(1)$. In this way, the number of ones in $\sigma^n(1)$ is a classical

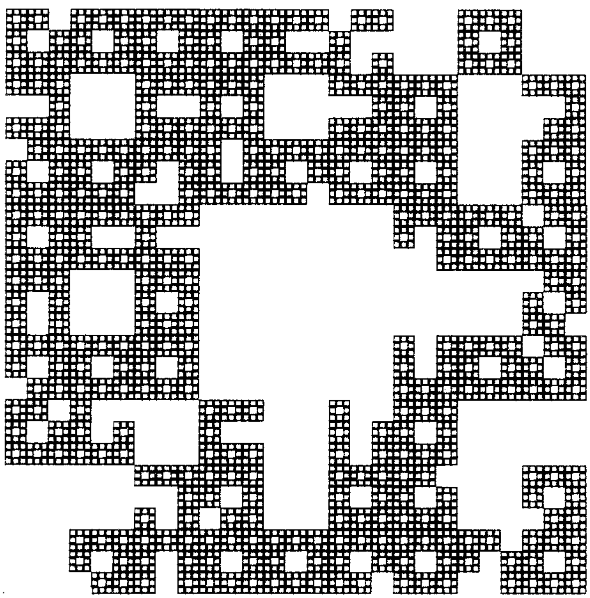

Fig. 1. A realization of A_4 for $p=0.9$.

Galton–Watson branching process with offspring distribution given by the p_i'. The limiting set A is called a random Cantor set.

Example 1.1 (Random Sierpinski carpet). Take $N=3$, $r=2$, $p_2=1-p_1=p\in[0,1]$, and

$$U_1=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},\qquad U_2=\begin{pmatrix}1&1&1\\1&0&1\\1&1&1\end{pmatrix}$$

Figure 1 shows a realization of A_4 for $p=0.9$.

We call a random substitution σ a *Bernoulli* (random) substitution with parameter p if there is a set J of indices (k,l) such that $\sigma(1)_{kl}=0$ if $(k,l)\notin J$ and $\mathbb{P}[\sigma(1)_{kl}=1]=1-\mathbb{P}[\sigma(1)_{kl}=0]=p$, for $(k,l)\in J$, independent of all other entries of $\sigma(1)$.

Example 1.2 (Mandelbrot percolation). Here the set A^σ is generated by a Bernoulli substitution σ with J the full set of indices. Note that $\sigma(1)$ takes values in the full set of 2^{N^2} matrices with entries 0 and 1. It is sometimes more practical to consider the random substitution $\tilde{\sigma}$ with $\tilde{\sigma}(0)=\mathbf{0}$ and (for $N=3$)

$$\mathbb{P}\left[\tilde{\sigma}(1)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\right]=1-\mathbb{P}\left[\tilde{\sigma}(1)=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\right]=1-p$$

Note that $A_{n+1}^{\tilde{\sigma}}=A_n^{\sigma}$ with probability p and $A_{n+1}^{\tilde{\sigma}}=\mathbf{0}$ with probability $1-p$. Hence, if $A^{\tilde{\sigma}}$ is not empty, then it is equal to A^σ. In the same way, the random Sierpinski carpet is generated by a Bernoulli substitution with

$$J=\{(k,l)\colon 1\leqslant k,l\leqslant 3,\ (k,l)\neq(2,2)\}$$

2. MORPHOLOGY OF RANDOM CANTOR SETS

Let A be a random Cantor set in the unit square. We shall classify different states of A, ordered according to increasing "denseness." Some of these are phrased in terms of the projection $\pi(A)$ of A onto the x axis and its Lebesgue measure $\lambda(\pi(A))$.

I. $A=\varnothing$ almost surely.

II. $\mathbb{P}[A=\varnothing]>0$, but $\dim(\pi A)=\dim(A)$.

III. $\dim(\pi A)<\dim(A)$ a.s. given $A\neq\varnothing$, but $\lambda(\pi A)=0$ a.s.

IV. $0 < \lambda(\pi(A)) < 1$ a.s. given $A \neq \varnothing$.

V. $\mathbb{P}[\lambda(\pi(A)) = 1] > 0$, but A does not percolate a.s.

VI. A percolates with positive probability.

Here $\dim(\cdot)$ denotes the Hausdorff dimension of a set, and A percolates means that A contains a connected set which has a nonempty intersection with the left and the right side of the unit square. As we will show, the random Sierpinski carpet (Example 1.1) passes through all of the six phases I,..., VI as the Bernoulli parameter p increases from 0 to 1.

We first mention that it is known when a random Cantor set is in one of the first three phases. Let m_l be the average number of ones in the lth column of $\sigma(1)$, i.e.,

$$m_l = \sum_{i=1}^{r} p_i \sum_{k=1}^{N} U_i(k, l)$$

Theorem 2.1. Let A be a random Cantor set. Then:

(i) $A = \varnothing$ a.s. iff $\sum_{l=1}^{N} m_l \leqslant 1$ [unless $\sigma(1)$ contains exactly one 1 a.s.].

(ii) $\dim(\pi A) = \dim(A)$ iff $\sum_{l=1}^{N} m_l \log m_l \leqslant 0$.

(iii) $\lambda(\pi A) = 0$ iff $\sum_{l=1}^{N} \log m_l \leqslant 0$.

Proof. (i) This follows from the extinction criterion for branching processes: $\sum_{l=1}^{N} m_l$ is equal to the average number of ones in $\sigma(1)$, which is the mean of the offspring distribution of the branching process whose nth generation is the number of 1's in $\sigma^n(1)$.

(ii) This is partly proved in Dekking and Grimmett,[3] completed in Falconer.[5]

(iii) This is proved in Dekking and Grimmett.[3] ∎

Example 2.2. (i) For the random Sierpinski carpet we have $m_1 = m_3 = 3p$ and $m_2 = 2p$. Hence $A = \varnothing$ a.s. for $0 \leqslant p \leqslant 1/8$, A is in phase II for $1/8 < p \leqslant 54^{-1/4} \approx 0.369$, and is in phase III for $54^{-1/4} < p \leqslant 18^{-1/3} \approx 0.381$.

(ii) For Mandelbrot percolation, there is no phase III: at $p = 1/3$, A passes from II to IV. Let $A = A(p)$ be a parametrized random Cantor set as in the examples above, and let $J(\alpha) = \{p: A(p) \text{ is in phase } \alpha\}$, for $\alpha = \text{I},\ldots,\text{VI}$. If for $\alpha < \beta$, $J(\alpha) \neq \varnothing$, $J(\beta) \neq \varnothing$, and $J(\gamma) = \varnothing$ for all $\alpha < \gamma < \beta$, then we define $p_{\alpha,\beta} = \sup J(\alpha) = \inf J(\beta)$. E.g., for the random set generated by the substitutions $\sigma(0) = \mathbf{0}$, $\sigma(1) = \left(\begin{smallmatrix}1 & 1\\ 0 & 0\end{smallmatrix}\right)$ with probability $1 - p$, and $\sigma(1) = \left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right)$ with probability p, only $p_{\text{V,VI}}$ is defined (and one can show that $1/2 < p_{\text{V,VI}} < 1$).

The theorem above shows that $p_{\mathrm{I,II}}$, $p_{\mathrm{II,III}}$, and $p_{\mathrm{III,IV}}$ can be exactly determined from the sole knowledge of the average number of 1's in each column of $\sigma(1)$. There is no hope for such simple criteria for the other critical values: phases V and VI involve interactions in horizontal directions. The best we can do is to show that all phases exist in the random Sierpinski carpet. We have not been able to prove the existence of phase V in the Mandelbrot percolation process.

The next two sections are devoted to an analysis of the existence of the phases in the Sierpinski carpet. In the final section we analyze the percolation phase in both the Sierpinski and the Mandelbrot processes.

3. MULTIVALUED AND RANDOM SUBSTITUTIONS

In this section we consider random substitutions without reference to geometrical interpretations as in the previous section. As the results in this section do not depend on whether the symbols are ordered in words or matrices, we will consider the first case (for notational convenience), but it will be obvious that they apply to the random Sierpinski carpet and Mandelbrot's percolation process.

We will now define a special class of multivalued substitutions (called $0L$-systems in Rozenberg and Salomaa[8]). Let $\Phi(1)$ be a nonempty set of words of length N, such that $0\cdots 0 \notin \Phi(1)$. Let $\Phi(0)$ be the complement of $\Phi(1)$ in $\{0, 1\}^N$. For words w and w', ww' denotes the concatenation of w and w'. Furthermore, we define the following sets:

$$\Phi(vw) = \{v'w \mid v' \in \Phi(v),\, w' \in \Phi(w)\}$$

$$\Phi^n(1) = \{\Phi(w) \mid w \in \Phi^{n-1}(1)\}, \qquad n \geqslant 2$$

The set $\Phi^n(0)$ is defined analogously. Note that Φ has the property that $\Phi(v) \cap \Phi(w) = \varnothing$ if $v \neq w$. This clearly implies $\Phi^n(0) \cap \Phi^n(1) = \varnothing$, for all n. It also follows by induction that $\Phi^n(0) \cup \Phi^n(1) = \{0, 1\}^{N^n}$. These observations prove the following lemma.

Lemma 3.1. $\Phi^n(0) = \{0, 1\}^{N^n} \backslash \Phi^n(1)$, for all $n \geqslant 1$.

Example 3.2. Let $N = 2$, $\Phi(1) = \{01\}$. Then $\Phi^2(1) = \{0001, 1001, 1101\}$.

Next we introduce randomness. Consider a Bernoulli random substitution with parameter p, which we denote by σ_p. Our principal object of study is

$$\pi_n(p) = \mathbb{P}[\sigma_p^n(1) \in \Phi^n(1)], \qquad n \geqslant 1$$

where $\Phi^n(1)$ is defined as above. To illustrate this, we first give an example.

Example 3.2 (continued). $N=2$, $J=\{1,2\}$, $\Phi(1)=\{01\}$. Now it follows from simple calculations that $\pi_1(p)=p(1-p)$, $\pi_2(p)=p^2(1-p)[1-p^2+p^3]=p^2(1-p)[1-p^2(1-p)]$, which is equal to $\pi_1(p^2(1-p))=\pi_1(p\pi_1(p))$.

The last equality is no coincidence, as the following proposition expresses.

Proposition 3.3. Let σ_p be a Bernoulli random substitution and let π_n be defined as above. Then we have, for all $n\geqslant 0$,

$$\pi_{n+1}(p)=\pi_1(p\pi_n(p))$$

where $\pi_0(p)$ is defined to be 1.

Proof. Let σ_p have index set J. We write

$$\begin{aligned}\Phi^{n+1}(1)&=\Phi^n(\Phi(1))\\&=\{\Phi^n(w_1)\,\Phi^n(w_2)\cdots\Phi^n(w_N)\mid w_1\cdots w_N\in\Phi(1)\}\end{aligned}$$

Consider $\sigma_p^{n+1}(1)$. This word lies in $\Phi^{n+1}(1)$ iff there is a $w_1\cdots w_N\in\Phi(1)$ such that $\sigma_p^{n+1}(1)\in\Phi^n(w_1)\cdots\Phi^n(w_N)$. This word $w=w_1\cdots w_N$ is called a starting point and according to Lemma 3.1 it is unique if it exists. But $\sigma_p^{n+1}(1)$ itself can be written as

$$\sigma_p^{n+1}(1)=\sigma_p^n(\sigma_p(1))=\sigma_p^n(u_1)\cdots\sigma_p^n(u_N)$$

where $u_i=0$ or 1 for all i. So $\sigma_p^{n+1}(1)\in\Phi^{n+1}(1)$ iff $\sigma_p^n(u_i)\in\Phi^n(w_i)$ for all i and for a unique starting point w. Note that $\sigma_p^n(u_1),\dots,\sigma_p^n(u_N)$ and $\sigma_p(1)=u_1\cdots u_N$ are independent. Now fix $w\in\Phi(1)$. We calculate the probability that w serves as starting point for the random word $\sigma_p^{n+1}(1)$, i.e., $\sigma_p^n(u_i)\in\Phi^n(w_i)$, $i=1,\dots,N$. We need only to look at indices in the set J. Suppose $i\in J$ and $w_i=1$. Then for $\sigma_p^n(u_i)$ to be in $\Phi^n(w_i)$, u_i must be 1 and $\sigma_p^n(1)\in\Phi^n(1)$. This obviously has probability $p\pi_n(p)$. Now suppose $i\in J$ and $w_i=0$. Then either (i) $u_i=0$ or (ii) $u_i=1$ and $\sigma_p^n(1)\in\Phi^n(0)$. According to Lemma 3.1, this is the complement of the event associated with $w_i=1$ and hence it has probability $1-p\pi_n(p)$. So the probability that w serves as starting point for $\sigma_p^{n+1}(1)$ equals $\mathbb{P}[\sigma_{p\pi_n(p)}(1)=w]$. Because $\sigma_p^{n+1}(1)\in\Phi^{n+1}(1)$ iff there is such a starting point $w\in\Phi(1)$, we obtain $\pi_{n+1}(p)=\pi_1(p\pi_n(p))$. ∎

It is not difficult to see from the definitions that $(\pi_n(p))$ is a non-increasing sequence in n. Hence the limit $\pi(p)=\pi^{\Phi}(p):=\lim_{n\to\infty}\pi_n(p)$ exists. We will be interested in the question of whether or not this limit is positive. To answer this question partially, we restrict ourselves to a special class of sets $\Phi(1)$. Consider a Bernoulli random substitution with index set J and a set of words $\Phi(1)$ as above. The set $\Phi(1)$ is said to be *increasing* if the following is true: if $w = w_1 \cdots w_N \in \Phi(1)$ and $w_i = 0$ for some $i \in J$, then $w' = w_1 \cdots w_{i-1} 1 w_{i+1} \cdots w_N \in \Phi(1)$. If $\Phi(1)$ is increasing, it follows from Grimmett (ref. 6, Section 2.5) that $\pi_1(p)$ is an increasing function in p. We define an iteration function G_p as follows:

$$G_p(x) = \pi_1(px), \qquad x, p \in [0, 1]$$

It follows from Proposition 3.3 that $\pi_{n+1}(p) = G_p(\pi_n(p))$ and we obtain

$$\pi_n(p) = G_p^n(1), \qquad n \geqslant 1$$

If $\Phi(1)$ is increasing, $G_p(\cdot)$ is increasing and hence $\pi(p)$ is equal to the largest fixed point of G_p. As said before, we are interested in the positivity of $\pi(p)$. For this we define

$$p_c(\Phi) = \inf\{p \mid \pi^{\Phi}(p) > 0\}$$

The following lemma tells us how to find $p_c(\Phi)$ if $\Phi(1)$ is increasing and in case of a Bernoulli random substitution.

Lemma 3.4. Consider a Bernoulli random substitution and let $\Phi(1)$ be an increasing set. Suppose that $(\partial/\partial p)\, \pi_1(p)|_{p=1} < 1$. Then $p_c(\Phi) < 1$ and $p_c(\Phi)$ is equal to the smallest p for which the following system $(*)$ has a solution:

$$(*) \begin{cases} G_p(x) & = x \\ \dfrac{\partial}{\partial x}\, G_p(x) = 1 \end{cases}$$

Proof. $G_p(x)$ is increasing in both p and x. Furthermore, $G_0(x) \equiv 0$ and $G_1(x) = \pi_1(x)$. If $G_1'(1) < 1$, then there is a $p < 1$ such that $G_p(x) = x$ for some $x > 1$, and hence $p_c(\Phi) < 1$. To find $p_c(\Phi)$, we now only have to look for the smallest p such that $G_p(x)$ is tangent to the line $y = x$. This is exactly the assertion of the lemma. ∎

The technique in this section generalizes the method of Chayes *et al.*,[1] which they use to prove that the percolation phase exists in the Mandelbrot percolation process. Applications of Lemma 3.4 will be given in the next section.

4. EXISTENCE OF ALL PHASES IN THE RANDOM SIERPINSKI CARPET

We will prove the following two lemmas.

Lemma 4.1. $1/2 < p_{\mathrm{IV,V}} \leqslant 0.8085$.

Lemma 4.2. $0.812 \leqslant p_{\mathrm{V,VI}} \leqslant 0.991$.

Together with Example 2.2(i), this is enough to show the existence of all phases. We remark that Lemma 4.2 implies that the critical value for the percolation phase in the 3×3 Mandelbrot process is also smaller then 0.991. This is a slight improvement over the results of Chayes *et al.*[1] The lower bounds in the two lemmas are proved with branching process techniques, while the upper bounds are obtained with the technique of the previous section, choosing proper sets $\Phi(1)$. In the following, a level-n square I^n_{kl} is said to be *open* if $\sigma^n_p(1)(k, l) = 1$; otherwise it is *closed.*

Proof of Lemma 4.1. The first inequality is easy. Let Z_n denote the number of open level-n squares in the middle column. Then Z_n is an ordinary branching process with mean offspring $2p$, where p is the Bernoulli parameter of the process. So for $p \leqslant 1/2$, Z_n will not survive. We need only to remark that if $Z_n = 0$ for some n, then $\lambda(\pi(A)) \leqslant 1 - 3^{-n}$. This proves the first inequality.

For the second inequality, let $\Phi(1)$ be the set of 0–1-valued 3×3 matrices such that each column contains at least one 1. It is obvious that if $\pi^{\Phi}(p) > 0$, then $p \geqslant p_{\mathrm{IV,V}}$. In the terminology of the previous section, we have

$$G_p(x) = p^3x^3(3 - 3px + p^2x^2)^2\,(2 - px) \tag{1}$$

$$\begin{aligned} x\frac{\partial}{\partial x} G_p(x) = {} & p^3x^3(3 - 3px + p^2x^2) \\ & \times (4px - 6)(5px - 2p^2x^2 - 3) \end{aligned} \tag{2}$$

From (1), we get, using $(*)$ of Lemma 3.4,

$$p^3x^3 = x\{(3 - 3px + p^2x^2)(2 - px)\}^{-1} \tag{3}$$

and substituting this in (2) yields after some calculations

$$-7p^3x^3 + 27p^2x^2 - 33px + 12 = 0$$

It follows that $px \cong 0.65524$. From (3), we get that $x \cong 0.8104$. Hence $p \cong 0.8085 = p_c(\Phi)$. This proves that $p_{\mathrm{IV,V}} \leqslant 0.8085$. ∎

Proof of Lemma 4.2. This requires a bit more work. For the first inequality, let S be a level-n open square in the middle column, which has open left and right neighbors (note that open level-n squares in the middle column never have upper or lower open neighbors). Let S_1 and S_2 be the two level-$(n+1)$ squares in the middle column contained in S. See Fig. 2 for a definition of $S_3, ..., S_6$.

We say that S_1 is *passable* if (i) S_3 and S_1 are both open and (ii) the right side of S_1 can be connected to the right side of either S_4 or S_6 or both, by open level-$(n+1)$ squares at the right side of S_1. For S_2, a similar definition holds, where S_3 and S_1 are replaced by S_5 and S_2, respectively.

Now, for $n \geqslant 2$, let Z_n denote the number of passable level-n squares. Then Z_n is an ordinary branching process. If this process becomes extinct, there cannot be percolation, so it is worthwhile to calculate its offspring distribution. Let the number of children of a particular passable square be denoted by M. Then M equals zero, one, or two. In Fig. 3 it is shown how $M=1$ can occur. The shaded squares are open, "c" means closed, the squares with an "a" are not allowed to be *all* open, and a "d" means that these squares are not allowed to be all open, but at least one of them is open.

The probabilities that the given configurations occur are $p^4(1-p^3)$, $p^6(1-p)^2$, $p^6(1-p)(1-p^2)$, and $2p^7(1-p)^2$, respectively. The first three configurations should in addition be reflected in $y=1/2$. We now obtain that the probability that $M=1$ equals $2p^4(1-p^3)+2p^6(1-p)^2(2+p)+2p^7(1-p)^2=2p^4(1-p)(1+p+3p^2-2p^4)$. Analogously, we find that $M=2$ with probability $p^8(3-2p)$. Hence we obtain $\mathbb{E}_p M = 2p^4(1+2p-3p^3+p^4)$. It follows that $\mathbb{E}_p M \leqslant 1$ iff $p \leqslant 0.81256$. Hence, for $p \leqslant 0.81256$ there is no percolation with probability one.

We remark that it is possible to improve this bound by also considering squares at the left side of S_1 and S_2. But for the existence of all phases this is not necessary and we omit this slight improvement.

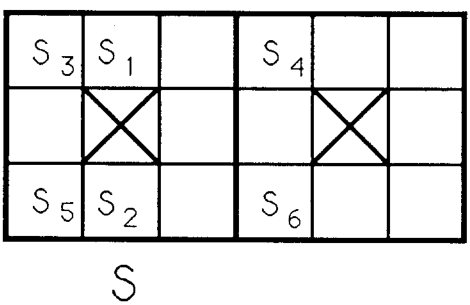

Fig. 2. The squares S, $S_1, ..., S_6$.

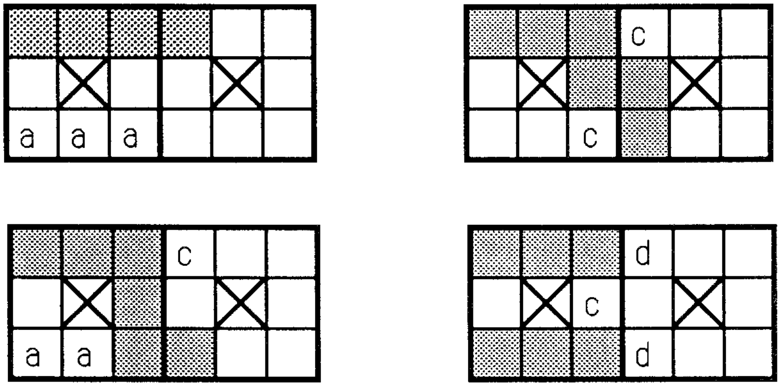

Fig. 3. Realizations of $M = 1$.

For the second inequality, we again use the technique of the previous section. Let $\Phi(1)$ be the (increasing) set of 3×3 matrices U with $U(2, 2) = 0$ and which contains at least seven ones. It is not difficult to prove by induction that if $p \geqslant p_c(\Phi)$, then $p \geqslant p_{\mathrm{V,VI}}$. To calculate $p_c(\Phi)$, we find, in the notation of the previous section,

$$G_p(x) = p^7 x^7 (8 - 7px) \tag{4}$$

$$x \frac{\partial}{\partial x} G_p(x) = 56 p^7 x^7 (1 - px) \tag{5}$$

Now $(*)$ from Lemma 3.4 easily leads to $px = 48/49$ and substituting this in $G_p(x) = x$ yields $x = (8/7)(48/49)^7$ and hence $p_c(\Phi) = (7/8)(49/48)^6 \cong 0.99023$. This proves Lemma 4.2. ∎

5. THE PERCOLATION PHASE

In this section we will analyze the sixth phase, in which there is a positive probability to have percolation. This phase exists in both Mandelbrot's percolation process and the random Sierpinski carpet, as we showed in the previous section. We remark here that if we set $p = 1$ for the Mandelbrot process, then it fills $[0, 1]^2$. However, percolation is not caused by this, because, for $p = 1$, the Sierpinski carpet is still a nowhere-dense, Lebesgue measure zero set.

Our main results (Theorem 5.3 and 5.4) have to do with the way in which the transition to the percolation phase takes place. In case of Mandelbrot's percolation process, these results were already obtained by

Chayes *et al.*[1] However, our proofs are much simpler and easily extend to the random Sierpinski carpet, where their method breaks down. In particular, we do not need the full analogue of the classical RSW theorem for ordinary percolation. Instead, we use a qualitative analogue which is easy to prove (see Lemma 5.1). We start with some notation. In this section σ_p stands for the Mandelbrot 3×3 substitution with Bernoulli parameter p, and $\tilde{\sigma}_p$ refers to the random Sierpinski carpet. For S a rectangle and σ a random substitution, $\mathscr{H}^{\sigma}(S)$ denotes the event that $A^{\sigma}\cap S$ contains a connected component which intersects the left and right sides of S. We say that S is crossed horizontally in that case. Similarly, we define $\mathscr{V}^{\sigma}(S)$, with "left" and "right" replaced by "top" and "bottom," respectively, and S is said to be crossed vertically in this case. Two principal objects of study are the percolation function θ and the critical value p_c defined by

$$\theta(p)=\mathbb{P}[\mathscr{H}^{\sigma_p}([0,1]^2)]$$

$$p_c=\inf\{p\mid\theta(p)>0\}$$

We define $\tilde{\theta}(p)$ and $\tilde{p}_c$ in the same way. It will turn out to be useful to consider the process in the rectangle $[0,1]\times[0,2]$, putting two independent copies of the process on top of each other. The corresponding definitions are

$$\tau(p)=\mathbb{P}[\mathscr{H}^{\sigma_p}([0,1]\times[0,2])]$$

and $\tilde{\tau}(p)$, which is defined using $\tilde{\sigma}_p$ instead of σ_p. It is immediate to see that $\tau(p)\geqslant\theta(p)$ and $\tilde{\tau}(p)\geqslant\tilde{\theta}(p)$. The following two lemmas are less clear as far as a proof is concerned. We think that no one will be surprised by their conclusions and we postpone the proofs till the end of this section.

Lemma 5.1. If $\tau(p)>0$, then $\theta(p)>0$.

Lemma 5.2. If $\tilde{\tau}(p)>0$, then $\tilde{\theta}(p)>0$.

Given these lemmas, short and elementary proofs can be given for a number of results concerning the phase transition to the percolation phase. The first asserts that the limiting set A^{σ_p} or $A^{\tilde{\sigma}_p}$ has a more violent phase transition than possibly expected from the bare definitions.

Theorem 5.3. If $\mathbb{P}[A^{\sigma_p}$ contains a connected component larger than one point$]>0$, then $\theta(p)>0$. The same is true if we replace σ_p and $\theta(p)$ by $\tilde{\sigma}(p)$ and $\tilde{\theta}(p)$, respectively.

Proof. We give the proof for the Mandelbrot percolation process. The proof in the case of the random Sierpinski carpet is identical. It follows from the assumption that for n large enough, there exists a level-n column

K_n, say, such that $\mathbb{P}[\mathcal{H}^{\sigma_p}(K_n)] > 0$. The column K_n contains the level-n squares $S_1, ..., S_{3^n}$. Now consider a connected component $C \subset A^{\sigma_p} \cap K_n$ such that C crosses K_n horizontally. Such a C exists with positive probability and there are two possibilities, at least one of which must have positive probability to occur:

1. C has a nonempty intersection with at least three level-n squares S_{i-1}, S_i, and S_{i+1}.
2. C has a nonempty intersection with at most two level-n squares S_j and S_{j+1}.

In case 1, it follows that $\mathbb{P}[\mathcal{V}^{\sigma_p}(S_i)] > 0$. But $\mathbb{P}[\mathcal{V}^{\sigma_p}(S_i)] = \mathbb{P}[\mathcal{H}^{\sigma_p}(S_i)] = p^n \theta(p)$ and hence $\theta(p) > 0$.

In case 2, we obtain $\mathbb{P}[\mathcal{H}^{\sigma_p}(S_j \cup S_{j+1})] > 0$. From this it is easy to see that either $\theta(p) > 0$ or $\tau(p) > 0$. According to Lemma 5.1, we obtain $\theta(p) > 0$ anyway. ▮

The following natural question comes to mind: what happens with the percolation functions θ and $\tilde{\theta}$ at p_c resp. $\tilde{p}_c$? In ordinary two-dimensional percolation, it is known that the corresponding percolation function is continuous (see, e.g., Grimmett[6]). In particular, the probability to have percolation at the critical point equals zero. There is a simple argument which shows that our processes both have *discontinuous* phase transitions, i.e., $\theta(p_c) > 0$ and $\tilde{\theta}(\tilde{p}_c) > 0$. Chayes *et al.*[1] claim that the reason for this discontinuity is the apparent asymmetry between open and closed squares: closed squares will always be closed, while open squares may be closed at any time. In our opinion, this is not the main reason for the discontinuity. The explanation lies in the rescaling property of the processes. As we will see in the proof of Theorem 5.4, the probability to cross a column horizontally is of the same order as the percolation probability. But to have percolation, three level-1 columns must be crossed horizontally. Because all these events have probabilities of the same order, this cannot be true if the percolation probability becomes too small. We will make this argument rigorous in the next proof.

Theorem 5.4. θ is discontinuous at p_c and $\tilde{\theta}$ is discontinuous at $\tilde{p}_c$.

Proof. Again we only prove the assertion for the Mandelbrot percolation process. The first remark is that it is an easy analytical fact that θ and τ are right-continuous (see, e.g., Chayes *et al.*[1]). It now follows from Lemma 5.1 that it is sufficient to prove that τ is discontinuous at p_c. It might seem strange, but this is much easier than a direct proof of the theorem.

If $\mathscr{H}^{\sigma_p}([0,1]\times[0,2])$ occurs, at least one of the following events must occur:

$$E_1^p = \bigcup_{i=1}^{6} \mathscr{V}^{\sigma_p}(S_i)$$

$$E_2^p = \bigcup_{i=1}^{5} \mathscr{H}^{\sigma_p}(S_i \cup S_{i+1})$$

where $S_1,..., S_6$ denote the six level-1 squares in a particular level-1 column, ordered from top to bottom. To see this, compare with the proof of Theorem 5.3: E_1 corresponds to case 1 and E_2 corresponds to case 2. Since there are three level-1 columns, we obtain

$$\tau(p) \leqslant \{\mathbb{P}[E_1^p \cup E_2^p]\}^3 \leqslant \{\mathbb{P}[E_1^p] + \mathbb{P}[E_2^p]\}^3$$

We have

$$\mathbb{P}[E_1^p] \leqslant 6\mathbb{P}[\mathscr{V}^{\sigma_p}(S_1)] = 6p\theta(p)$$

and

$$\begin{aligned}\mathbb{P}[E_2^p] &\leqslant 5\mathbb{P}[\mathscr{H}^{\sigma_p}(S_1 \cup S_2)] \\ &= 5[p^2\tau(p) + 2p(1-p)\,\theta(p)] \\ &\leqslant 5p^2\tau(p) + 10p\theta(p)\end{aligned}$$

Combining these estimates, we obtain

$$\tau(p)^{1/3} \leqslant 16p\theta(p) + 5p^2\tau(p) \leqslant 21\tau(p)$$

Now suppose $\tau(p) \neq 0$. Then it follows that

$$\tau(p)^{2/3} \geqslant 1/21$$

or

$$\tau(p) \geqslant (1/21)^{3/2}$$

Thus, τ is discontinuous at p_c. ∎

This leaves us with the proofs of Lemmas 5.1 and 5.2. Their proofs are not at all difficult, but require some definitions. The proof of Lemma 5.1 can be seen as a qualitative analogue to the classical RSW theorem from ordinary percolation. It shows that if $\mathbb{P}[\mathscr{H}^{\sigma_p}([a,b]\times[c,d])] > 0$, then also $\mathbb{P}[\mathscr{H}^{\sigma_p}([a,e]\times[c,d])] > 0$ for all $e > b$. The only nonelementary ingredient of the proof is the well-known FKG inequality (see, e.g., Grimmett[6]). It asserts that if E_1 and E_2 are two increasing or two decreasing events which depend on at most countably many random variables (as is the case in our model), then $\mathbb{P}[E_1 \cap E_2] \geqslant \mathbb{P}[E_1]\,\mathbb{P}[E_2]$.

Proof of Lemma 5.1. We need some definitions. Let S_1^n denote a level-n square in $[0, 3^{-n}] \times [0, 2]$ (i.e., in the first level-n column), S_2^n a level-n square in the ith level-n column, for some $i \in \{1,..., 3^{n-1}\}$, T_1^m a level-m square in the jth column for some $j \in \{2 \cdot 3^{m-1} + 1,..., 3^m\}$, and T_2^m a level-m square in the final level-m column. The rows containing these squares are denoted by s_1^n, s_2^n, t_1^m, and t_2^m. Note that S_1^n and S_2^n are squares in $[0, 1/3] \times [0, 2]$, while T_1^m and T_2^m are contained in $[2/3, 1] \times [0, 2]$. We consider the following event:

$C(S_1^n, S_2^n, T_1^m, T_2^m) = \{A^{\sigma_p}$ contains a connected component C which crosses $[0, 1] \times [0, 2]$ horizontally, such that the intersection of C with the first i level-n columns connects the left side of S_1^n to the right side of S_2^n, and the intersection of C with the final $3^m - j + 1$ level-m columns connects the left side of T_1^m to the right side of $T_2^m\}$

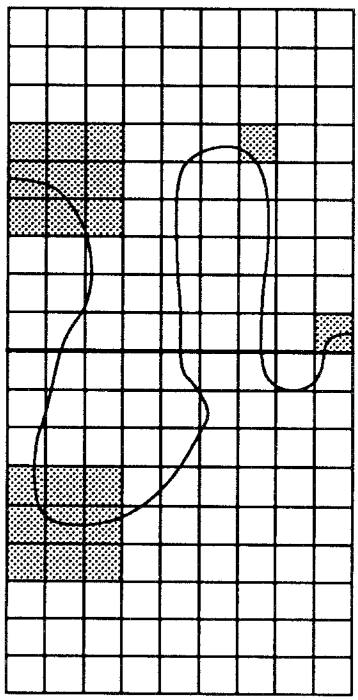

Fig. 4. The event $C(S_1^1, S_2^1, T_1^2, T_2^2)$.

Now $\tau(p) > 0$, so for each choice of n and m we can find S_1^n, S_2^n, T_1^m, and T_2^m as above with $\mathbb{P}[C(S_1^n, S_2^n, T_1^m, T_2^m)] > 0$. Suppose that this is true for a choice S_1^n, S_2^n, T_1^m, and T_2^m with $s_1^n \neq s_2^n$ and $t_1^m \neq t_2^m$; see Fig. 4, where $n = 1$, $m = 2$, and where the squares S_1^1, S_2^1, T_1^2, and T_2^2 are shaded.

Given this event, Fig. 5 shows how to build a crossing of $[0, 3] \times [0, 2]$. Rather than giving a turgid formal proof, we remark that it follows from reflection, translation invariance, and the FKG inequality that such a crossing has positive probability. It then follows that $\mathbb{P}[\mathscr{H}^{\sigma_p}([0, 3] \times [0, 3])] > 0$ and hence, by rescaling, $\theta(p) > 0$.

It remains to prove the existence of squares S_1^n, S_2^n, T_1^m, and T_2^m with these properties. First we only require $s_1^n \neq s_2^n$, and suppose such squares do not exist. This implies that there is a level-2 row i_2, say, such that there is a positive probability $q > 0$, say, that a horizontal crossing of $[0, 1] \times [0, 2]$ intersects the first three level-2 squares of this row. In particular, all first three level-2 squares of this row must be open. Furthermore, there is a level-3 subrow of i_2, i_3 say, such that there is a probability of at least $q3^{-1}$ that this crossing intersects the first nine squares of this row. Again, these first nine squares must be open in that case. Iterating this, we conclude that the probability that there exists a level-n row with all its first 3^{n-1} level-n squares open is at least $q3^{2-n}$. On the other hand, it is easy to deduce that this probability is at most

$$2 \cdot 3^n \mathbb{P}[\text{row 1 has all its first } 3^{n-1} \text{ squares open}] \leqslant 2 \cdot 3^n p^{3^{n-1}}$$

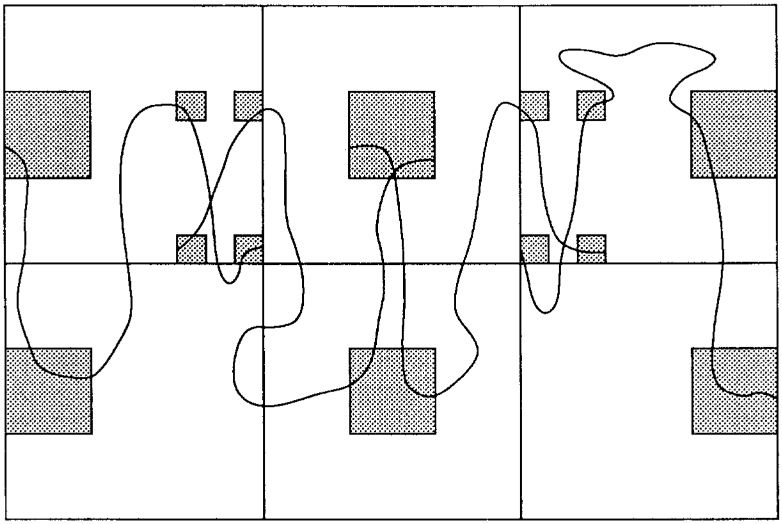

Fig. 5. Building a crossing of $[0, 3] \times [0, 2]$.

Hence

$$q3^{2-n} \leqslant 2 \cdot 3^n p^{3^{n-1}}$$

which is impossible for n large enough. Now fix S_1^n and S_2^n, $s_1^n \neq s_2^n$, with

$$\mathbb{P}[C(S_1^n, S_2^n, T_1^m, T_2^m)] > 0$$

Using the same argument, it follows that we can find T_1^m and T_2^m, $t_1^m \neq t_2^m$, such that this event still has positive probability. ∎

Proof of Lemma 5.2. The proof above does not work here because of the lack of translation invariance. But because of the special structure of this process, a proof is even easier. Suppose $\tilde{\theta}(p) = 0$. Let K_n denote the middle column in $[0, 1] \times [0, 2]$ after n subdivisions. It follows by construction that K_n can contain only two adjacent open level-n squares for all n. Furthermore, the probability that there do exist two such squares goes down exponentially fast; in fact, it equals p^{2n}. Now let Z denote the first index n such that K_n does not contain two such squares. Then Z is finite almost surely and hence

$$\begin{aligned}\tilde{\tau}(p_,) &\leqslant \sum_{n=1}^{\infty} \mathbb{P}[Z=n]\ \mathbb{P}[\mathscr{H}^{\tilde{\sigma}_p}(K_n) \mid Z=n] \\ &\leqslant \sum_{n=1}^{\infty} \mathbb{P}[Z=n]\{1-[1-\tilde{\theta}(p)]^{2n}\} = 0\end{aligned}$$

which is a contradiction. Hence $\tilde{\theta}(p) > 0$. ∎

We end with a remark concerning the Mandelbrot process. We considered in this section the 3×3 case. Nothing changes essentially if we replace 3 by a larger integer. For the 2×2 case, however, a minor problem arises to show the existence of the percolation phase. It is not possible to find a proper increasing set $\Phi(1)$ with the property that, if $p \geqslant p_c^{\Phi}$, then $p \geqslant p_{\mathrm{V,VI}}$. We can solve this problem by comparing with the 4×4 case, as remarked in Chayes *et al.*[1] Using simple coupling, it is not difficult to show that if p_i denotes $p_{\mathrm{V,VI}}$ in the $i \times i$ Mandelbrot process, it is true that $p_2 \leqslant 1-(1-p_4^{1/2})^4$, which is strictly smaller than one.

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